Contents lists available at Science-Gate

International Journal of Advanced and Applied Sciences

Journal homepage: http://www.science-gate.com/IJAAS.html

A new transformation technique to find the analytical solution of general second order linear ordinary differential equation

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ARTICLE INFO

Article history:
Received 17 November 2017
Received in revised form
16 February 2018
Accepted 25 February 2018

Keywords:
Ordinary differential equations
Analytical solution
Transformations

ABSTRACT

The purpose of this paper is to introduce a new analytical approach towards the general solution of ordinary linear differential equations (OLDEs) of order two. The method involves a transformation based on integral function in an exponential form which leads to the general solution of given differential equation. A special case of second order OLDEs has been discussed to develop the formulae and solution procedure and different problems have been solved to explain the solution method. Finally, the idea has been extended to solve the general form of second order OLDEs.

1. Introduction

A large class of problems in physics and engineering are modelled in the form of differential equations which may be linear or nonlinear. And a large number of these problems involve second order OLDEs i.e. oscillations, damped motion, resonance, L-R-C circuits as discussed by Zill (2016). The general form of second order homogeneous OLDE is

$$P(x)u'' + Q(x)u' + R(x)u = 0 \quad (1)$$

where $u = h(x)$ is the solution of Eq. 1 and the coefficients $P(x)$, $Q(x)$ and $R(x)$ are the functions of x and $P(x) \neq 0$. The existence of solution and the method to find the solution of OLDE depends upon the nature of these coefficient functions $P(x)$, $Q(x)$ and $R(x)$, see Saravi (2012). Some of the OLDEs can be easily handled and their closed form analytical solutions can be obtained by means of classical methods. But it is not always possible to solve Eq. 1 in its general form. We try to formulate a method of solution by narrowing down the scope of these coefficient functions. For example, if we choose $P(x)$, $Q(x)$ and $R(x)$ as constant functions then it is very easy to transform the differential Eq. 1 to its corresponding algebraic equation by applying the transformation $u = e^{mx}$, see Zill (2016). The roots of algebraic equation corresponding to Eq. 1 yield general solution to Eq. 1 by applying traditional methods. Similarly, if we restrict the coefficient functions of Eq. 1 as $P(x) = x^2$, $Q(x) = x$and $R(x) = 1$ continuous over the interval $(0, \infty)$ then Eq. 1 becomes well known Cauchy-Euler equation which can also be solved by transformation method by using the transformation $u = x^m$, see Zill (2016). In the same way, many attempts have been made to transform Eq. 1 into simple forms to make it solvable with the help of different transformations i.e. Laplace transformation, theory of natural transformation, Aboodh transformation, Adomian decomposition, operator factorization, analytical extension and many other discussed by Geisbauer (2007), Hasan (2012), Kim (2016), Mohammed and Zeleke (2015), Robin (2007), Saravi (2012), Wilmer III and Costa (2008), Johnson et al. (2008), and Belgacem and Silambarasan (2013).

The summudu transformation has also proved its importance in engineering mathematics and problem solving. It may be used to solve problems without resorting them into a new frequency domain. Belgacem and Karaballi (2006), Belgacem (2006), Katatbeh and Belgacem (2011), Belgacem et al. (2003), and Belgacem and Silambarasan (2017) made theoretical investigations to establish fundamental and advanced properties in various aspects of summudu transform. Due to mathematical simplicity and effectiveness, the summudu transformation has been applied to solve Maxwell's equations for transient electromagnetic waves propagation and Bessel equation, see Hussain and Belgacern (2007), Belgacem and Silambarasan (2012), Belgacem (2010). Summudu transformation is equally well applicable to nonlinear, fractional and partial differential equations, see Touchent and

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https://doi.org/10.21833/ijaas.2018.04.014

Belgacem (2015) and Belgacem and Karaballi (2006).

On the other hand when we talk about nonlinear differential equations, the Riccati differential equation possesses great interest and importance. The general form of Riccati equation is

$$u' = a(x) + b(x)u + c(x)u^2 \tag{2}$$

where $a(x)$, $b(x)$ and $c(x)$ are real valued scalar functions of x, see Schwabl (2005). More interestingly, the Riccati equation can be transformed into second order OLDE by using the transformation $y = -\frac{u'}{a(x)u}$ as discussed by Busawon and Johnson (2005). The study made by Busawon and Johnson (2005) and Pala and Ertas (2017) showed that Eq. 1 and 2 are inter-convertible. The Riccati equations are modelled in many fields of quantum mechanics such as Schrödinger equation in quantum mechanics see Schwabl (2005) particularly, in quantum chemistry, see Fraga et al. (1999), industrial control and optimization problems, stability analysis of state estimation Cai et al. (2017), Kalman filtering Li et al. (2015), power electronics and navigation Dehghannasiri et al. (2017), Ahmeid et al. (2017), and the Gross-Pitaevskii equation (GPE) which occurs in Bose-Einstein condensates (BECs), see Yuce and Kilic (2006). These all problems involve Riccati differential equation or OLDEs of order two; see Al Bastami et al. (2010). Many attempts have also been made to solve Riccati Eq. 2 in the form of special cases in the literature Al Bastami et al. (2010), Batiha (2015), Busawon and Johnson (2005), Pala and Ertas (2017), and Sarafian (2011) but once again there is no general method to solve (2) in its general form Al Bastami et al. (2010).

All the above literature and discussion shows the importance of Eqs. 1 and 2 for their applications but at the same time we can observe that every method to find the general solution of these equations has its own limitations and specific area of implementation based on the nature of coefficient functions, see Al Bastami et al. (2010), Batiha (2015), Busawon and Johnson (2005), Johnson et al. (2008), Kim (2016), Mohammed and Zeleke (2015), Pala and Ertas (2017), Robin (2007), Sarafian (2011), Saravi (2012), and Wilmer III and Costa (2008). In the next section, we have discussed the special case of second order OLDE on the basis of a new transformation to obtain its general solution. We have modified the transformation used by (Pala and Ertas, 2017) to solve second order OLDEs.

2. Special case I

Consider the differential equation

$$u'' + Q(x)u' + R(x)u = 0 \tag{3}$$

which is the special case of Eq. 1 showing $P(x) = 1$. Let us consider the coefficient functions $Q(x)$ and $R(x)$ as piecewise continuous functions of x over the real interval $I \subseteq \mathbb{R}$.

Theorem 2.1: The non-trivial solution $u(x)$ to the differential Eq. 3 satisfying $R(x) = \frac{f''}{f}$, where $f = ce^{\int \frac{Q(x)}{2}dx}$ is non-zero real valued piecewise continuous function of x over the interval $I_0 \subseteq \mathbb{R}$ and arbitrary constants c, c_1, c_2 can be written as $u(x) = c_1 e^{-\int \frac{Q(x)}{2}dx} + c_2 x\, e^{-\int \frac{Q(x)}{2}dx}$.

Proof: Consider the transformation

$$\tilde{u}(x) = f\, e^{\int \frac{g\,u'(x)}{u(x)}dx} \tag{4}$$

where g is a functions of $x$continuous over the interval $I_0 \subseteq \mathbb{R}$ and $\tilde{u}(x)$ is the transformed function corresponding to $u(x)$. In the following text we use u and $\tilde{u}$ instead of $u(x)$ and $\tilde{u}(x)$ respectively. Differentiate Eq. 4 w.r.t x to get following equations

$$\begin{aligned}\tilde{u}'(x) &= e^{\int \frac{gu'}{u}dx}\left[\frac{fgu'}{u} + f'\right]\\ \tilde{u}'' &= e^{\int \frac{gu'}{u}dx}\left[\frac{fgu''}{u} - \frac{fgu'^2}{u^2} + \frac{fgu'}{u} + \frac{f'gu'}{u} + f'' + \frac{fg^2u'^2}{u^2} + \frac{f'gu'}{u}\right]\end{aligned} \tag{5}$$

we rearrange Eq. 5 to get

$$\frac{u}{fg}e^{-\int \frac{gu'}{u}dx}\,\tilde{u}'' + \frac{(1-g)u'^2}{u} = u'' + \left[\frac{2f'}{f} + \frac{g'}{g}\right]u' + \frac{f''u}{fg}. \tag{6}$$

By comparing Eqs. 3 and 6 we get,

$$\begin{cases} Q(x) = \frac{2f'}{f} + \frac{g'}{g}, \\ R(x) = \frac{f''}{fg}, \\ \frac{u}{fg}e^{-\int \frac{gu'}{u}dx}\,\tilde{u}'' + \frac{(1-g)u'^2}{u} = 0. \end{cases} \tag{7}$$

Since u is a non-trivial solution therefore $u, u' \neq 0$ and since f and g are non-zero functions of x therefore $\frac{u}{fg}e^{-\int \frac{gu'}{u}dx} \neq 0$. By our assumption as a special case if $(1-g) = 0$then $\frac{u}{fg}e^{-\int \frac{gu'}{u}dx}\,\tilde{u}'' + \frac{(1-g)u'^2}{u} = 0$ in Eq. 7 Implies

$$\begin{cases} \tilde{u}'' = 0, \\ g = 1 \end{cases} \tag{8}$$

Using Eq. 7 the set of equations in (7) become

$$\begin{cases} Q(x) = \frac{2f'}{f}, \\ R(x) = \frac{f''}{f}. \end{cases} \tag{9}$$

By integrating (8) w.r.t x twice, we get

$$\tilde{u} = a_0 x + b_0 \tag{10}$$

where a_0 and b_0 are arbitrary constants. By comparing Eqs. 4 and 10 we get

$$f\, e^{\int \frac{gu'}{u}dx} = a_0 x + b_0\ ,$$

By substituting the values and simplification we get

$$u = \frac{a_0 x + b_0}{f}. \quad (11)$$

Using the equation $Q(x) = \frac{2f'}{f}$ from (9) we get

$$f = ce^{\int \frac{Q(x)}{2} dx} \quad (12)$$

where c is constant of integration and $Q(x)$ and $P(x)$ are mutually connected as given in (9). Using f form (12) in (11) we get

$$u = c_1 e^{-\int \frac{Q(x)}{2} dx} + c_2 x\, e^{-\int \frac{Q(x)}{2} dx} \quad (13)$$

where c_1 and c_2 are arbitrary constants such that $c_1 = \frac{b_0}{c}$ and $c_2 = \frac{a_0}{c}$.

Eq. 13 gives the general solution of Eq. 3 containing two linearly independent functions such that Wronskian,

$$W\left(e^{-\int \frac{Q(x)}{2} dx},\ x\, e^{-\int \frac{Q(x)}{2} dx}\right) \neq 0.$$

Now we illustrate above stated method with the help of examples given below.

Example 2.1: Consider the differential equation $u'' + 2u' + u = 0$. On comparison with general Eq. 3 we get $Q(x) = 2$ and $R(x) = 1$. Eq. 12 implies that $f = ce^x$ satisfying set of equations in (9). So, we can write general solution by using Eq. 13 as $u = c_1 e^x + c_2 x\, e^x$.

Example 2.2: We consider another differential equation $u'' + 5xu' + \left(\frac{25}{4}x^2 + \frac{5}{2}\right)u = 0$. On comparison with general Eq. 3 we get $Q(x) = 5x$ and $R(x) = \frac{25}{4}x^2 + \frac{5}{2}$. Eq. 12 implies that $f = ce^{\frac{5}{4}x^2}$ satisfying (9). Eq. 13 yields the general solution as $u = c_1 e^{-\frac{5}{4}x^2} + c_2 x\, e^{-\frac{5}{4}x^2}$.

3. Special case II

Consider the differential equation

$$Pu'' + (PQ - P')u' + (P)^2 R\, u = 0 \quad (14)$$

which is a special case of Eq. 1 having P, Q and R as piecewise continuous functions of x over the real interval $I \subseteq \mathbb{R}$.

Theorem 3.1: The non-trivial solution to the differential Eq. 14 satisfying $R = \frac{f''}{f\,P}$, where $f = \frac{c' e^{\int \frac{Q}{2} dx}}{\sqrt{P}}$ and P are non-zero real valued piecewise continuous functions of x over the interval $I_0 \subseteq \mathbb{R}$ can be written as $u = c_3\sqrt{P}e^{-\int \frac{Q}{2} dx} + c_4\sqrt{P}x\, e^{-\int \frac{Q}{2} dx}$, where c', c_3, c_4 are arbitrary constants.

Proof: Consider the transformation

$$\tilde{u} = f\, e^{\int \frac{g u'}{Pu} dx}. \quad (15)$$

Differentiate Eq. 15 w.r.t x twice

$$\tilde{u}' = e^{\int \frac{gu'}{Pu} dx}\left[\frac{fgu'}{Pu} + f'\right]$$
$$\tilde{u}'' = e^{\int \frac{gu'}{Pu} dx}\left[\frac{fgu''}{Pu} + \frac{f'gu'}{Pu} + \frac{f'gu'}{Pu} - \frac{fgu'^2}{Pu^2} - \frac{fgP'u'}{P^2u} + f'' + \frac{fg^2u'^2}{P^2u^2} + \frac{f'gu'}{Pu}\right] \quad (16)$$

Multiply (16) by $e^{-\int \frac{gu'}{Pu} dx}\frac{P^2u}{fg}$ and rearrange to get

$$\frac{P^2u}{fg}\, e^{-\int \frac{gu'}{Pu} dx}\, \tilde{u}'' + (P - g)\frac{u'^2}{u} = Pu'' + \left(2\frac{Pf'}{f} - P' + \frac{Pg'}{g}\right)u' + \frac{f''P^2}{fg}u \quad (17)$$

By comparing (17) with (14) we get,

$$\begin{cases} Q = \frac{2f'}{f} + \frac{g'}{g}, \\ R = \frac{f''}{fg}, \\ \frac{P^2u}{fg}\ e^{-\int \frac{gu'}{Pu} dx}\, \tilde{u}'' + (P - g)\frac{u'^2}{u} = 0 \end{cases} \quad (18)$$

Since u is a non-trivial solution therefore $u, u' \neq 0$ and since P, f and g are non-zero functions of x therefore $\frac{P^2u}{fg}\ e^{-\int \frac{gu'}{Pu} dx} \neq 0$. By our assumption as a special case if $(P - g) = 0$then $\frac{P^2u}{fg}\ e^{-\int \frac{gu'}{Pu} dx}\, \tilde{u}'' + (P - g)\frac{u'^2}{u} = 0$ in (18) Implies

$$\begin{cases} \tilde{u}'' = 0, \\ g = P. \end{cases} \quad (19)$$

Using (19) the set of equations in (18) become

$$\begin{cases} Q(x) = \frac{2f'}{f} + \frac{P'}{P}, \\ R(x) = \frac{f''}{f\,P}. \end{cases} \quad (20)$$

Integrating $\tilde{u}'' = 0$ in (19) w.r.t x twice we get

$$\tilde{u} = a_1 x + b_1 \quad (21)$$

where a_1 and b_1 are arbitrary constants. By comparing (15) and (21) we get

$$f\, e^{\int \frac{gu'}{Pu} dx} = a_1 x + b_1\,,$$

By substituting values and simplification we get

$$u = \frac{a_1 x + b_1}{f}. \quad (22)$$

Now we obtain the value of f from (18) as

$$f = \frac{c' e^{\int \frac{Q}{2} dx}}{\sqrt{P}} \quad (23)$$

where c' is constant of integration and P, Q and R are relationally connected as seen from (20). Substituting f from (23) in (22), we get

$$u = c_3\sqrt{P}e^{-\int \frac{Q}{2} dx} + c_4\sqrt{P}\, x\, e^{-\int \frac{Q}{2} dx} \quad (24)$$

where c_3 and c_4 are arbitrary constants such that

$$c_3 = \frac{b_1}{c'}$$

and

$$c_2 = \frac{a_1}{c'}.$$

Eq. 13 gives the general solution of (3) containing two linearly independent functions such that Wronskian,

$$W\left(\sqrt{P}e^{-\int\frac{Q}{2}dx}, \sqrt{P}\, x\, e^{-\int\frac{Q}{2}dx}\right) \neq 0.$$

To demonstrate the utility of above stated method we present the following examples.

Example 3.1: Consider the differential equation $6xu'' - 6u' - \frac{3}{2x}u = 0$. On comparison with general Eq. 14 we get $P = 6x$, $Q = \frac{2}{x}$ and $R = -\frac{1}{24x^3}$. Eq. 23 implies that $f = c'\sqrt{\frac{x}{6}}$ satisfying (18). So, (24) implies the general solution as $u = c_1\sqrt{x} + c_2\frac{1}{\sqrt{x}}$.

Example 3.2: Consider the problem

$$4u'' + 32xu' + 16\,(4x^2 - 3)u = 0 \tag{25}$$

On comparison with general Eq. 14 we get $P = 2$, $Q = 8x$ and $R = 4x^2 - 3$. Eq. 23 implies that $f = c'e^{2x^2}$ which do not satisfy the condition (18) namely $R \neq \frac{f''}{fg}$ where $g = P$. So the problem cannot be solved by using above stated method. For the further explanation and solution we discuss the general case in following text.

4. General form of second order OLDE

The transformation represented by Eq. 15 in special case II involves P which yields the general solution on conditional basis (i.e. Solution exists only if $f = \frac{c'e^{\int\frac{Q}{2}dx}}{\sqrt{g}}$ satisfies the condition $R = \frac{f''}{fg}$) and the method get restricted to the special case of general second order differential Eq. 1. In the next theorem we have reduced the condition $R = \frac{f''}{fg}$ so that this method can be worked in the larger spectrum.

Theorem 4.1: The non-trivial solution $u(x)$ to the differential Eq. 14 is given by $u(x) = \frac{c_5\,\tilde{u}}{f}$, where $f = \frac{c''e^{\int\frac{Q}{2}dx}}{\sqrt{P}}$ and $\tilde{u}$ can be obtained by solving $\tilde{u}'' - \left(\frac{f''}{f} - PR\right)\tilde{u} = 0$ where f, $P \neq 0$ are real valued piecewise continuous functions of x over the interval $I_0 \subseteq \mathbb{R}$ and c'', c_5 are arbitrary constants.

Proof: By using same transformation discussed in case II and Eq. 17 and imposing the condition

$$\frac{P^2u}{fg}\,\tilde{u}''\,e^{-\int\frac{gu'}{Pu}dx} + (P - g)\frac{u'^2}{u} = S\,u \tag{26}$$

where S is the function of x such that (14) becomes

$$Pu'' + (PQ - P')u' + (P)^2R\,u = S\,u. \tag{27}$$

If we take $g = P$ as given in (19) then (26) becomes

$$\frac{P^2u}{fg}\,\tilde{u}''\,e^{-\int\frac{gu'}{Pu}dx} = S\,u\ ,$$

or

$$\tilde{u}'' = \frac{g}{P^2}f\,e^{\int\frac{gu'}{Pu}dx}S. \tag{28}$$

Substituting (15) in (28) and by using $g = P$ as given in (19), we get

$$\tilde{u}'' - \frac{S}{P}\tilde{u} = 0. \tag{29}$$

If we choose

$$S = P^2\left(\frac{f''}{f\,P} - R\right) \tag{30}$$

where $\left(\frac{f''}{f\,P} - R\right)$ is a factor which makes equation solvable by removing the condition given in theorem 3.1. Using (30) in (29) we get

$$\tilde{u}'' - \left(\frac{f''}{f} - PR\right)\ \tilde{u} = 0. \tag{31}$$

The solution to (14) depends upon the solution of (31) which is very simple second order differential equation as compared to (14). We can solve (31) by classical methods to obtain the value of $\tilde{u}$. By using $g = P$ as given in (19), the Eq. 15 becomes

$$u = \frac{\tilde{u}}{f} \tag{32}$$

where f can be obtained from $Q(x) = \frac{2f'}{f} + \frac{P'}{P}$ given in (20) as

$$f = \frac{c''e^{\int\frac{Q}{2}dx}}{\sqrt{P}} \tag{33}$$

where c'' is constant of integration. By combining (32) and (33) we get the general solution as

$$u = c_5\tilde{u}\sqrt{P}e^{-\int\frac{Q}{2}dx} \tag{34}$$

where $c_5 = \frac{1}{c''}$ and $\tilde{u}$ is obtained by (31) contains two parameter family of solutions.

Example 4.1: We continue with the problem given in example 3.2. By using (30) we choose $S = 64$ and rearrange the problem to get

$$4u'' + 32xu' + 16\,(4x^2 + 1)u = 64u \tag{35}$$

Eq. 31 implies

$$\tilde{u}'' - 16\tilde{u} = 0. \tag{36}$$

By solving (36), we get $\tilde{u} = a_2 e^{4x} + b_2 e^{-4x}$. Thus (34) implies the general solution to (35) as

$$u = c_6 e^{-2x^2+4x} + c_7 e^{-2x^2-4x},$$

where c_6 and c_7 are arbitrary constants.

The literature in Shivakumar and Zhang (2013) gives the power series solution of $y'' - f(x)y = 0$ which may support us to solve (31).

Corollary 4.1: The general solution to the differential Eq. 3 can be obtained by using $P = 1$ in theorem 4.1 and given by the equation $u = \frac{c_5\,\tilde{u}}{f}$, where $f = c'' e^{\int \frac{Q}{2} dx}$ and $\tilde{u}$ can be obtained by solving $\tilde{u}'' - \left(\frac{f''}{f} - R\right)\tilde{u} = 0$ and $f \neq 0$ is real valued piecewise continuous function of x over the interval $I_0 \subseteq \mathbb{R}$ and c'', c_5 are arbitrary constants.

Example 4.2: Consider the differential equation

$$u'' + 4u' - (x-4)u = 0. \tag{37}$$

By comparing (37) with (3) we have $Q = 4$ and $R = (4 - x)$. According to corollary 4.1, $f = c'' e^{2x}$ and $\tilde{u}'' - \left(\frac{f''}{f} - R\right)\tilde{u} = 0$ implies

$$\tilde{u}'' - x\tilde{u} = 0. \tag{38}$$

Eq. 38 is an Airy differential equation with the solution $\tilde{u} = c_6 Ai(x) + c_7 Bi(x)$, where

$$Ai(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{2\cdot 3\cdot 5\cdot 6\cdots(3k-3)\cdot(3k-1)\cdot 3k}$$

and

$$Bi(x) = x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{3\cdot 4\cdot 6\cdot 7\cdots(3k-2)\cdot 3k\cdot(3k+1)}.$$

Hence the general solution to (37) can be written as $u = c_6 e^{-2x} Ai(x) + c_7 e^{-2x} Bi(x)$, where c_6 and c_7 are arbitrary constants.

5. Conclusion

The main idea discussed in this paper is to investigate and develop a method of solution to OLDEs of order two by using transformation technique. The role of coefficient functions of second order OLDE in the solution process has also been discussed to develop the solution procedure. We started working with the special case of second order OLDE and proved the solution formula and its procedure through examples. This provided us the basis to extend the idea for general cases by removing conditions on coefficient functions and we developed new solution formulae and the working procedures to make this technique applicable to general problems. We hope that our work would be useful for the researchers in the field of differential equations, advanced engineering and applied sciences.

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